

MAS 117

Introduction to Algebra

Solutions

Assignment 8

1 (a) $\mathbb{N}$ fails the additive inverse axiom (A3): there is no natural number n such that $n + 2 = 0$.

(b) $2\mathbb{Z} + 1$ fails the additive closure axiom (A0): 1 and 3 are odd integers but $1 + 3$ is not. [It also fails the additive identity axiom (A2): there is no odd integer z such that $a + z = a$ for all odd integers a .]

(c) The set of invertible 2×2 real matrices fails the additive closure axiom (A0): I_2 and $-I_2$ are invertible but their sum $O_{2 \times 2}$ is not. [It also fails the additive identity axiom.]

(d) The set of polynomials in which the coefficient of x^2 is zero fails the multiplicative closure axiom (M0): the coefficient of x^2 in the polynomials $x + 1$ and $x + 2$ are both zero but the coefficient of x^2 in their product is 1.

(e) The set of vectors in real 3-dimensional space fails the multiplicative associative law (M1), as we saw in Chapter 2 of the notes.

2 (a) Follow the proof that $0a = 0$ given in lectures:

$$a0 + a0 = a(0 + 0) = a0 = a0 + 0,$$

and the cancellation law gives the result.

(b) If $1 = 0$ then, for any element $a \in R$, we have

$$a = a1 = a0 = 0,$$

so $R = \{0\}$, contradicting the fact that R has more than one element.

(c) Suppose that 0 has a multiplicative inverse z . Then

$$1 = 0z = 0,$$

contradicting part (b).

3 (a) We have to show that $\sim$ is reflexive, symmetric and transitive.

reflexive: 1 is a unit, and $a = a1$, so $a \sim a$ for any $a \in R$.

symmetric: If $b = au$, where u is a unit with inverse u^{-1} , then $a = a1 = auu^{-1} = bu^{-1}$, and u^{-1} is a unit, so $b \sim a$.

transitive: Suppose that $a \sim b$ and $b \sim c$; say $b = au$ and $c = bv$ where u and v are units. Then $c = a(uv)$, and uv is a unit; so $a \sim c$.

(b) In $\mathbb{Z}_{12}$, we saw that the units are the elements $[a]_{12}$, where $\gcd(a, 12) = 1$; that is, 1, 5, 7 and 11 (we write a instead of $[a]_{12}$ from now on).

The equivalence class of a is the set of all elements au , as u runs through the set of units; that is, it is $a, 5a, 7a, 11a$. So the equivalence classes are:

- $\{0\}$
- $\{1, 5, 7, 11\}$
- $\{2, 10\}$
- $\{3, 9\}$
- $\{4, 8\}$
- $\{6\}$